

Incompressible, quasi-rigid deformations of 2-dimensional domains

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Abstract

This paper proposes a sensible definition of a deformation metric between 2-dimensional surfaces obtained from each other by an area preserving (incompressible) mapping, and an algorithm for obtaining this metric, as well as the optimal deformation.

1 Introduction

Recently, there is an increasing interest in the analysis of near-rigid deformation within the computer vision community, in particular pattern recognition, image segmentation and face recognition ([B2K], [EK], [MS]).

For example, a variety of objects can be represented as point clouds. These can be obtained by sampling of the objects in question by points in some canonical Euclidian space. One is often presented with the problem of deciding whether two of these point clouds, and/or the corresponding underlying objects or manifolds, represent the same geometric structure (object recognition and classification).

To quantify the difference between two such clouds it is natural to construct smooth domains out of the samplings, and look for a mapping from one domain to the other which is "as close to an isometry as possible". The minimal deviation of these mappings from a rigid deformation (isometry) can stand as a measure of similarity between the original objects.

However, this task is very difficult from a computational point of view, since the set of all mappings between two domains is very large. On the other hand, it is sensible to assume that the *density* of the sample points reflects the true nature of the object. This implies that the *volume elements* associated with these domains, created out of the samplings, are prescribed. In particular we may restrict ourselves to an approximation of an isometry which preserve the volume of the two domains.

The object of this paper is to propose a sensible definition of a deformation metric between 2-dimensional surfaces obtained from each other by incompressible (in this case, area preserving) mapping, and an algorithm for obtaining this metric, as well as the optimal deformation.

In section 2 the problem is formulated for a pair of flat domains. Section 3 introduces the analytic conditions for quasi-rigid deformation, for flat domains. In section 4 the problem is extended to a pair of embedded surfaces in \mathbb{R}^3 . In section 5 we use the results of the previous sections to propose an algorithm, namely a flow which converges (formally) to a quasi rigid deformation. For the convenience of the reader we defer all technical proofs to the final section 6.

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2 A particular example

Consider, for example, two flat domains $\Omega, \Omega_1 \subset \mathbb{R}^2$, equipped with the Euclidian metric $e(dx, dy) = dx^2 + dy^2$. If Ω_1 is a rigid deformation of Ω then there exists an isometry $\Phi : (\Omega, e) \rightarrow (\Omega_1, e)$. Locally, it means

$$|\Phi_x|^2 \equiv |\Phi_y|^2 \equiv 1, \quad \Phi_x \cdot \Phi_y \equiv 0 \quad \text{on } \Omega. \quad (2.1)$$

In general Ω and Ω_1 are not isometric, and there is no mapping verifying (2.1). However, Φ is, *by definition*, an isometry between (Ω, g_Φ) and (Ω_1, e) where

$$g_\Phi(dx, dy) := \begin{pmatrix} dx & dy \end{pmatrix} |D\Phi|^2 \begin{pmatrix} dx \\ dy \end{pmatrix} \equiv |\Phi_x|^2 dx^2 + 2(\Phi_x \cdot \Phi_y) dx dy + |\Phi_y|^2 dy^2. \quad (2.2)$$

$$|D\Phi|^2 := \begin{pmatrix} |\Phi_x|^2 & \Phi_x \cdot \Phi_y \\ \Phi_x \cdot \Phi_y & |\Phi_y|^2 \end{pmatrix}_{(x,y)}.$$

How can we quantify the deviation of (Ω_1, e) from an isometric image of (Ω, e) ? Certainly, it is related to the deviation of g_Φ from the Euclidian metric. Recall that g_Φ is represented by a symmetric 2×2 matrix. So, we consider a real valued function defined on the set of symmetric 2×2 real matrices $S(2; \mathbb{R})$. Let $h_\alpha : S(2; \mathbb{R}) \rightarrow \mathbb{R}$, where α is some real parameter (see below), verifying

$$h_\alpha(A) = h_\alpha(UAU^*) \quad \text{for any } U \in O(2; \mathbb{R}) \text{ and } A \in S(2; \mathbb{R}),$$

$$h_\alpha(A) \geq h_\alpha(I) \equiv 0$$

where I is the identity 2×2 matrix, and the equality holds if and only if $A = I$. Now set

$$H_\alpha(\Omega, \Omega_1) = \inf_{\Phi \in Diff(\Omega; \Omega_1)} \int_{\Omega} h_\alpha(|D\Phi|^2) dx dy. \quad (2.3)$$

By this definition we obtain that (Ω, e) and (Ω_1, e) are isometric if and only if $H_\alpha(\Omega, \Omega_1) = 0$.

What is a natural choice of h_α ? It must be a function of the eigenvalues λ_i , $i = 1, 2$ of A , hence it depends on only 2 arguments, say $tr(A)$ and $det(A)$. Note that

$$k(A) := tr^2(A) - 4det(A) = (\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 = (\lambda_1 - \lambda_2)^2 \geq 0, \quad (2.4)$$

and $A = I$ if and only if $k(A) = 0$ and $det(A) = 1$. This leads us to the natural choice

$$h_\alpha(A) = k(A) + \alpha (det(A) - 1)^2,$$

where $\alpha > 0$ is a parameter.

In this paper we restrict ourselves to incompressible deformations. This corresponds to the choice $\alpha = \infty$ which implies the constraint $det(A) = 1$. The adaptation of the quasi-rigid deformation metric (2.3) to the incompressible case is obtained by the *constrained optimization*

$$H_\infty(\Omega, \Omega_1) := \inf_{\Phi \in O(\Omega; \Omega_1)} \int_{\Omega} k(|D\Phi|^2) dx dy \quad (2.5)$$

where $O(\Omega; \Omega_1)$ is the set of all *area preserving* diffeomorphisms $\Phi : \Omega \rightarrow \Omega_1$, that is,

$$O(\Omega, \Omega_1) := \{ \Phi \in Diff(\Omega; \Omega_1) ; \quad |\Phi_x|^2 |\Phi_y|^2 - (\Phi_x \cdot \Phi_y)^2 \equiv 1 \text{ on } \Omega \} . \quad (2.6)$$

Can we compare the domains Ω, Ω_1 using the definition (2.5)? Evidently, $H_\infty(\Omega; \Omega_1) < \infty$ if and only if the set $O(\Omega; \Omega_1)$ is non empty. By its definition, a necessary condition is

$$Area(\Omega) = Area(\Omega_1) . \quad (2.7)$$

By a theorem of Moser [M] it turns out that condition (2.7) is also sufficient, under rather general conditions.

Assuming (2.7), the existence of a minimizer of (2.5) is a much more difficult problem. By the definition (2.4) of $k(\cdot)$ and (2.6) we pose the following equivalent open problem.

Open problem: *Suppose (2.7). Is there a minimizer of*

$$\inf_{\Phi \in O(\Omega; \Omega_1)} \hat{H}(\Phi) \quad \text{where} \quad \hat{H}(\Phi) := \int_{\Omega} tr^2(|D\Phi|^2) dx dy \quad ? \quad (2.8)$$

3 Main result for flat domain

The set (2.6) is, formally, an infinite dimensional manifold. There is an associated right-translation on this manifold by the group of area preserving diffeomorphisms

$$O(\Omega) := \{ S \in Diff(\Omega) ; \quad |S_x|^2 |S_y|^2 - (S_x \cdot S_y)^2 \equiv 1 \text{ on } \Omega \} . \quad (3.1)$$

Indeed, $O(\Omega)$ is a group under composition, and its action on $O(\Omega; \Omega_1)$ from the right is defined by

$$\Phi \in O(\Omega; \Omega_1) \implies \Phi \circ S \in O(\Omega; \Omega_1) .$$

$O(\Omega)$ is a formal Lie group. Its Lie algebra is given by

$$o(\Omega) := \{ \mathbf{v} \in C^\infty(\Omega; \mathbb{R}^2) \mid \nabla \cdot \mathbf{v} \equiv 0 \text{ on } \Omega, \quad \mathbf{v} \cdot \hat{n} \equiv 0 \text{ on } \partial\Omega \} .$$

Any smooth flow $t \rightarrow \Phi^{(t)} \in O(\Omega; \Omega_1)$ is generated by an orbit $t \rightarrow \mathbf{v}^{(t)} \in o(\Omega)$ via

$$\frac{d}{dt} \Phi^{(t)} = \mathbf{v}^{(t)} \left(\Phi^{(t)} \right) . \quad (3.2)$$

The flow (3.2) can be presented by the Euler equation

$$\frac{\partial}{\partial t} \Phi^{(t)} - \mathbf{v}^{(t)} \cdot D\Phi^{(t)} = 0 . \quad (3.3)$$

Our object is to define $\mathbf{v}^{(t)} \in o(\Omega)$ for which $\hat{H}(\Phi^{(t)})$ is strictly decreasing, and is stationary if and only if $\Phi^{(t)}$ is a critical point of \hat{H} in the sense described below.

At this stage it is convenient to consider complex number notation. Here we represent $(x, y) \sim x + iy \in \mathbb{Z}$. Then

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y) ; \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y), \quad dz = dx + idy, \quad d\bar{z} = dx - idy, \quad dz \wedge d\bar{z} \sim -2idxdy . \quad (3.4)$$

Lemma 3.1 below is elementary from the divergence theorem and the definition (3.4):

Lemma 3.1. *If Ω is simply connected then for any $\mathbf{v} \in o(\Omega)$ there exists a function $\psi \in C^\infty(\Omega)$ so that $\psi \equiv 0$ on $\partial\Omega$ and $\mathbf{v} = (1/2)(-\psi_y, \psi_x)$ on Ω . In particular, $\mathbf{v} \sim v_1 + iv_2 = i\partial_{\bar{z}}\psi$ and*

$$i\partial_{\bar{z}}\psi d\bar{z} = \frac{\partial\psi}{\partial n}|dz| \text{ on } \partial\Omega$$

where $\partial\psi/\partial n$ is the (real) outward normal derivative of ψ on $\partial\Omega$.

For $\Phi \in O(\Omega; \Omega_1)$, let the Hopf function [H]

$$f_\Phi(z, \bar{z}) := |\Phi_y|^2 - |\Phi_x|^2 + 2i(\Phi_x \cdot \Phi_y) . \quad (3.5)$$

The main result is:

Theorem 1. *Let $\Phi^0 \in O(\Omega; \Omega_1)$. Let $\Phi^{(t)}$ be a flow (3.3) where $\mathbf{v}^{(t)} \sim i\partial_{\bar{z}}\psi^{(t)}$ satisfying $\psi^{(t)} \in C^\infty(\Omega)$, $\psi^{(t)} = 0$ on $\partial\Omega$ for any $t \in \mathbb{R}$. Then $\Phi^{(t)} \in O(\Omega; \Omega_1)$ for any $t \in \mathbb{R}$ and*

$$\frac{d}{dt} \hat{H}(\Phi^{(t)}) = 4\Im \int_{\Omega} f_{\Phi^{(t)}} \partial_{\bar{z}}^2 \psi dx dy . \quad (3.6)$$

Now, we are in a position to define quasi-rigid deformation as a critical point of the functional (2.8) on the constraint manifold (2.6).

Definition 3.1. *A mapping $\Phi \in O(\Omega_1; \Omega)$ is quasi-rigid if and only if $d\hat{H}(\Phi)/dt = 0$ for any $\psi \in C^\infty(\Omega)$ verifying $\psi \equiv 0$ on $\partial\Omega$.*

Proposition 3.1. *$\Phi : \Omega \rightarrow \Omega_1$ is a quasi rigid deformation if and only if (2.7) and the following conditions hold:*

$$i) \Im(\partial_{\bar{z}}^2 f_\Phi) = 0 \text{ on } \Omega.$$

$$ii) \Im(f_\Phi \frac{dz}{d\bar{z}}) = 0 \text{ on } \partial\Omega. .$$

Remark 3.1. *It is interesting to compare the conditions of Proposition 3.1 to the case of harmonic maps between two Riemannian surfaces. In that case the function f_Φ is holomorphic, that is, $\partial_{\bar{z}} f_\Phi = 0$. This follows from the stationarity of the Dirichlet functional \hat{H} with respect to all parameterizations of the domain Ω . In the case under consideration, the functional \hat{H} is constrained to the set $O(\Omega)$, and the criticality condition (i) of the Proposition is weaker. Note also that (ii) can also be written as $\Im(f_\Phi dz^2) = 0$ on $\partial\Omega$, while $f_\Phi dz^2$ is the Hopf differential (see Definition 1.3.10 of [H]).*

4 Generalizations

Here we generalize the results of section 3. Instead of flat domains Ω, Ω_1 , we consider a pair of smooth, compact surfaces Σ, Σ_1 embedded in \mathbb{R}^3 .

Assume that Σ, Σ_1 are diffeomorphic to a canonical domain Δ . For simplicity we may think about the case $\Delta := \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$. We shall later comment about the case where Ω, Ω_1 are embedded manifolds without boundary (e.g. Δ is the unit sphere \mathbb{S}^2).

Let

$$\mathbf{X} : \Delta \rightarrow \Sigma = \mathbf{X}(\Delta) \subset \mathbb{R}^3, \quad \mathbf{Y} : \Delta \rightarrow \Sigma_1 = \mathbf{Y}(\Delta) \subset \mathbb{R}^3,$$

diffeomorphisms from Δ to Σ and Σ_1 (respectively), then the mapping

$$\mathbf{Y} \circ \mathbf{X}^{-1} : \Sigma \rightarrow \Sigma_1 \quad (4.1)$$

describes a diffeomorphism between the surfaces Σ and Σ_1 .

In order to investigate the deviation of $\mathbf{Y} \circ \mathbf{X}^{-1}$ from an isometry, we use the mappings \mathbf{X} and \mathbf{Y} to pull back the metrics from Σ and Σ_1 respectively, to Δ . The geometry of Σ is pulled back to Δ by the parameterization \mathbf{X} as

$$g_{\mathbf{X}}(dx, dy) := \begin{pmatrix} dx & dy \end{pmatrix} G_{\mathbf{X}}|_{(x,y)} \begin{pmatrix} dx \\ dy \end{pmatrix} \equiv \|\mathbf{X}_x\|^2 dx^2 + 2\langle \mathbf{X}_x \cdot \mathbf{X}_y \rangle dx dy + \|\mathbf{X}_y\|^2 dy^2. \quad (4.2)$$

where the standard inner product $\langle \cdot, \cdot \rangle$, $\|\mathbf{v}\|^2 := \langle \mathbf{v}, \mathbf{v} \rangle$ in \mathbb{R}^3 is used, and

$$G_{\mathbf{X}}|_{(x,y)} := \begin{pmatrix} \|\mathbf{X}_x\|^2 & \langle \mathbf{X}_x \cdot \mathbf{X}_y \rangle \\ \langle \mathbf{X}_x \cdot \mathbf{X}_y \rangle & \|\mathbf{X}_y\|^2 \end{pmatrix}_{(x,y)}.$$

The corresponding area element pulled back from Σ to Δ via \mathbf{X} is

$$|\mathbf{X}_x \wedge \mathbf{X}_y| dx dy = \det(G_{\mathbf{X}}) dx dy.$$

Similar expressions hold for the metric induced from Σ_1 on Δ via \mathbf{Y} .

Given such reference parameterizations \mathbf{X} and \mathbf{Y} , the set of *all* diffeomorphisms from Σ to Σ_1 can be represented by the set of all diffeomorphisms $\Phi : \Delta \rightarrow \Delta$ as follows:

$$\mathbf{Y} \circ \Phi^{-1} \circ \mathbf{X}^{-1} : \Sigma \rightarrow \Sigma_1, \quad \Phi \in \text{Diff}(\Delta). \quad (4.3)$$

Given $\Phi \equiv (\phi_1, \phi_2) \in \text{Diff}(\Delta)$ we inquire the deviation of the mapping (4.3) from an isometry. The pull-back of the metric from Σ to Δ by $\mathbf{X} \circ \Phi := \mathbf{X}^\Phi : \Delta \rightarrow \Sigma$ is given by

$$G_{\mathbf{X}^\Phi}|_{(x,y)} := D\Phi_{(x,y)} \circ G_{\mathbf{X}}|_{\Phi(x,y)} \circ D^*\Phi_{(x,y)} \quad (4.4)$$

where

$$D\Phi_{(x,y)} := \begin{pmatrix} \partial_x \phi_1 & \partial_x \phi_2 \\ \partial_y \phi_1 & \partial_y \phi_2 \end{pmatrix}, \quad D^*\Phi_{(x,y)} := \begin{pmatrix} \partial_x \phi_1 & \partial_y \phi_1 \\ \partial_x \phi_2 & \partial_y \phi_2 \end{pmatrix}.$$

Likewise, the area element from Σ is pulled back to this on Δ by

$$|\mathbf{X}_x^\Phi \wedge \mathbf{X}_y^\Phi|_{(x,y)} dx dy = \det(G_{\mathbf{X}}|_{\Phi(x,y)}) \det^2(D\Phi_{(x,y)}) dx dy.$$

In particular, for any $\Phi \in \text{Diff}(\Delta)$, the surface area $|\Sigma|$ of Σ is given by

$$|\Sigma| \equiv \int_{\Delta} \det(G_{\mathbf{X}}|_{\Phi(x,y)}) \det^2(D\Phi_{(x,y)}) dx dy.$$

and is independent of Φ . Lemma 4.1 below follows from definition:

Lemma 4.1. *The diffeomorphism (4.3) is an isometry between Σ and Σ_1 if and only if*

$$D\Phi_{(x,y)} \circ G_{\mathbf{X}}|_{\Phi(x,y)} \circ D^*\Phi_{(x,y)} = G_{\mathbf{Y}}|_{(x,y)} \quad \text{on } \Delta . \quad (4.5)$$

Likewise, (4.3) is area preserving if and only if

$$\det \left(G_{\mathbf{X}}|_{\Phi(x,y)} \right) \det^2 (D\Phi_{(x,y)}) = \det \left(G_{\mathbf{Y}}|_{(x,y)} \right) \quad \text{on } \Delta . \quad (4.6)$$

We now choose a particular reference parameterizations \mathbf{X}, \mathbf{Y} as follows. Recall that the *Uniformisation Theorem* [FK] implies that there exist *conformal* parameterizations of these surfaces. We may assume, therefore, that \mathbf{X} (res. \mathbf{Y}) are conformal parameterizations to Σ , (res. Σ_1). This means

$$G_{\mathbf{X}} \equiv \mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad G_{\mathbf{Y}} \equiv \eta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{on } \Delta , \quad (4.7)$$

where μ^2 (res. η^2) are the area densities on Δ associated with the conformal parameterizations of Σ (res. Σ_1). With these particular parameterizations, condition (4.5) for isometry of the mapping (4.3) is reduced to

$$\mu(\Phi(x,y)) |D\Phi|_{(x,y)}^2 = \eta(x,y) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{on } \Delta , \quad (4.8)$$

where $|D\Phi|^2 = D\Phi \circ D^*\Phi$. Likewise, (4.3) is area preserving if and only if

$$\mu(\Phi(x,y)) \det (D\Phi_{(x,y)}) = \eta(x,y) \quad \text{on } \Delta .$$

Let us consider the set

$$O(\Sigma, \Sigma_1) := \{ \Phi \in \text{Diff}(\Delta) ; \quad \mu(\Phi(x,y)) \det (D\Phi_{(x,y)}) = \eta(x,y) \quad \forall (x,y) \in \Delta \} . \quad (4.9)$$

Again, the result of Moser [M] implies that $O(\Sigma, \Sigma_1)$ is not empty, provided

$$|\Sigma| = |\Sigma_1| . \quad (4.10)$$

Now, we recall (2.4). It implies

$$k \left(D\Phi_{(x,y)} G_{\mathbf{X}}|_{\Phi(x,y)} D^*\Phi_{(x,y)} \right) \geq 0 \quad (4.11)$$

and, by Lemma 4.1, it follows that Σ, Σ_1 are isometric if and only if

$$k \left(D\Phi_{(x,y)} G_{\mathbf{X}}|_{\Phi(x,y)} D^*\Phi_{(x,y)} \right) \equiv 0 \quad \text{on } \Delta . \quad (4.12)$$

Under the assumption (4.7) we obtain

$$k \left(D\Phi_{(x,y)} G_{\mathbf{X}}|_{\Phi(x,y)} D^*\Phi_{(x,y)} \right) = \mu^2(\Phi(x,y)) \text{tr} \left(|D\Phi|_{(x,y)}^2 \right) - 4\mu^2(\Phi(x,y)) \det \left(|D\Phi|_{(x,y)}^2 \right) .$$

For $\Phi \in O(\Sigma, \Sigma_1)$ it follows

$$k \left(D\Phi_{(x,y)} G_{\mathbf{X}}|_{\Phi(x,y)} D^*\Phi_{(x,y)} \right) = \mu^2(\Phi(x,y)) \text{tr} \left(|D\Phi|_{(x,y)}^2 \right) - 4\eta^2(x,y) . \quad (4.13)$$

Using (4.11,4.12) and (4.13) we obtain

Lemma 4.2. Assume (4.10). If \mathbf{X}, \mathbf{Y} are conformal parameterizations and $\Phi \in O(\Sigma, \Sigma_1)$ then

$$\hat{H}_\Delta(\Phi) := \int_\Delta \mu^2(\Phi) \text{tr}(|D\Phi|^2) dx dy \geq 4|\Sigma_1| . \quad (4.14)$$

The equality in (4.14) holds if and only if Σ and Σ_1 are isometric.

We are now at a position to generalize (2.8): Calculate

$$\inf_{\Phi \in O(\Sigma, \Sigma_1)} \hat{H}_\Delta(\Phi) \quad (4.15)$$

and find the optimal mapping Φ (if exists).

As in section 3, the right action on the "manifold" $O(\Sigma, \Sigma_1)$ is given by the Lie group of η preserving diffeomorphisms on Δ :

$$O_\eta(\Delta) := \{S \in \text{Diff}(\Delta) ; \quad \eta(S(x, y)) \det(DS_{(x, y)}) = \eta(x, y) \quad \forall (x, y) \in \Delta . \} . \quad (4.16)$$

Lemma 4.3. The set $O_\eta(\Delta)$ is a group under compositions. For any $\Phi \in O(\Sigma, \Sigma_1)$ and $S \in O_\eta(\Delta)$, $\Phi \circ S \in O(\Sigma, \Sigma_1)$. In particular, $O_\eta(\Delta)$ acts on $O(\Sigma, \Sigma_1)$ by right-translations.

The Lie algebra associated with $O_\eta(\Delta)$ is

$$o_\eta := \{ \mathbf{v} \in C^\infty(\Delta; \mathbb{R}^2) \mid \nabla \cdot (\eta \mathbf{v}) \equiv 0 \text{ on } \Delta, \quad \mathbf{v} \cdot \hat{n} \equiv 0 \text{ on } \partial\Delta . \} . \quad (4.17)$$

In complex notation (3.4), Lemma 3.1 implies that

$$\mathbf{v} \in o_\eta \text{ if and only if } \mathbf{v} \sim v_1 + iv_2 = i\eta^{-1} \partial_{\bar{z}} \psi$$

where $\psi \in C^\infty(\Delta)$ and $\psi \equiv 0$ on $\partial\Delta$.

As in the case of planar domains discussed in section 3, we consider now the flow $\Phi^{(t)} \in O(\Sigma, \Sigma_1)$ generated by the Euler equation

$$\frac{\partial}{\partial t} \Phi^{(t)} - \mathbf{v}^{(t)} \cdot D\Phi^{(t)} = 0 \quad , \quad \mathbf{v}^{(t)} \in o_\eta , \quad (4.18)$$

and the Hopf function f_Φ defined, as in (3.5), on Δ . The generalization of Theorem 1 is now formulated as

Theorem 2. Let $\Phi^0 \in O(\Sigma, \Sigma_1)$. Let $\Phi^{(t)}$ be a flow (4.18) where $\mathbf{v}^{(t)} \sim i\eta^{-1} \partial_{\bar{z}} \psi^{(t)}$ satisfying $\psi^{(t)} \in C^\infty(\Omega)$, $\psi^{(t)} = 0$ on $\partial\Omega$ for any $t \in \mathbb{R}$. Then $\Phi^{(t)} \in O(\Sigma, \Sigma_1)$ for any $t \in \mathbb{R}$ and

$$\frac{d}{dt} \hat{H}_\Delta(\Phi^{(t)}) = 4\Im \int_\Delta \mu^2(\Phi^{(t)}) f_\Phi \partial_{\bar{z}} (\eta^{-1} \partial_{\bar{z}} \psi) dx dy . \quad (4.19)$$

Now, we generalize Definition 3.1 and Proposition 3.1 as follows:

Definition 4.1. A mapping $\mathbf{Q} : \Sigma \rightarrow \Sigma_1$ is quasi-rigid if and only if it can be decomposed as

$$\mathbf{Q} = \mathbf{Y} \Phi^{-1} \mathbf{X}^{-1} \quad (4.20)$$

where $\mathbf{X} : \Delta \rightarrow \Sigma$, $\mathbf{Y} : \Delta \rightarrow \Sigma_1$ are conformal diffeomorphisms and $\Phi \in O_\eta(\Delta)$ verifying $d\hat{H}_\Delta(\Phi)/dt = 0$ for any $\psi \in C^\infty(\Delta)$ for which $\psi \equiv 0$ on $\partial\Delta$.

Proposition 4.1. \mathbf{Q} given by (4.20) is a quasi rigid deformation verifying (4.7) if and only if the corresponding $\Phi \in O_\eta(\Delta)$ verifies

$$i) \quad \Im \partial_{\bar{z}} \{ \partial_{\bar{z}} (\mu^2(\Phi) f_\Phi) \eta^{-1} \} = 0 \quad \text{on } \Delta .$$

$$ii) \quad \Im (f_\Phi \frac{dz}{d\bar{z}}) = 0 \quad \text{on } \partial\Delta .$$

5 Applications

Theorem 1 in section 3 suggests an algorithm for calculating a quasi rigid deformations from a flat domain $\Omega \subset \mathbb{R}^2$ to another $\Omega_1 \subset \mathbb{R}^2$ where (2.7) is assumed.

Theorem 3. *Let $\Phi^{(0)} \in O(\Omega)$. Define the flow $\Phi^{(t)}$ in $O(\Omega)$ given by the Euler equation (3.3) for $t \geq 0$ and $\mathbf{v}^{(t)} \sim i\partial_{\bar{z}}\psi^{(t)} \in o(\Omega)$ is defined as follows: $\psi^{(t)} = \psi_0^{(t)} + \psi_1^{(t)} + \psi_2^{(t)}$ where*

$$\psi_0^{(t)} := -\Im(\partial_{\bar{z}}^2 f_{\Phi^{(t)}}) \quad (5.1)$$

on Ω and

$$\int_{\Omega} \psi_1^{(t)} \psi_0^{(t)} dx dy = \int_{\Omega} \psi_2^{(t)} \psi_0^{(t)} = 0$$

as well as

$$\psi_1^{(t)} = -\psi_0^{(t)} \text{ and } \partial_z \psi_1^{(t)} = -\partial_z \psi_0^{(t)} \text{ on } \partial\Omega$$

and

$$\psi_2^{(t)} = 0 \quad , \quad \frac{\partial \psi_2^{(t)}}{\partial n} = -\Im\left(f_{\Phi^{(t)}} \frac{dz}{d\bar{z}}\right) \text{ on } \partial\Omega \quad ,$$

then the flow (3.3) satisfies

$$\frac{d\hat{H}(\Phi^{(t)})}{dt} = -2 \int_{\Omega} \int |\Im(\partial_{\bar{z}}^2 f_{\Phi^{(t)}})|^2 dx dy - \int_{\partial\Omega} \left| \Im\left(f_{\Phi^{(t)}} \frac{dz}{d\bar{z}}\right) \right|^2 |dz| \quad .$$

In particular, $d\hat{H}(\Phi)/dt \leq 0$ and Φ is a steady state of (3.3) if and only if it is a quasi-rigid deformation.

Similarly, Theorem 2 in section 4 suggests an algorithm for calculating a quasi rigid deformations from an embedded surface $\Sigma \subset \mathbb{R}^3$ to another $\Sigma_1 \subset \mathbb{R}^3$:

Theorem 4. *Let $\Sigma \subset \mathbb{R}^3, \Sigma_1 \subset \mathbb{R}^3$ and a pair of conformal parameterizations \mathbf{X}, \mathbf{Y} verifying (4.7). Let $\Phi^{(0)} \in O_{\eta}(\Delta)$. Define the flow $\Phi^{(t)}$ in $O_{\eta}(\Delta)$ given by the Euler equation (4.18) for $t \geq 0$ and $\mathbf{v}^{(t)} \sim i\partial_{\bar{z}}\psi^{(t)} \in o_{\eta}$ is defined as in Theorem ??, where (5.1) is replaced by*

$$\psi_0^{(t)} = -\Im\partial_{\bar{z}} \left\{ \partial_{\bar{z}} \left(\mu^2(\Phi^{(t)}) f_{\Phi^{(t)}} \right) \eta^{-1} \right\} \quad . \quad (5.2)$$

then the flow (4.18) satisfies

$$\frac{d\hat{H}_{\Delta}(\Phi^{(t)})}{dt} = -2 \int_{\Delta} \left| \Im\partial_{\bar{z}} \left\{ \partial_{\bar{z}} \left(\mu^2(\Phi^{(t)}) f_{\Phi^{(t)}} \right) \eta^{-1} \right\} \right|^2 dx dy - \int_{\partial\Delta} \eta^{-1} \mu^2(\Phi^{(t)}) \left| \Im\left(f_{\Phi^{(t)}} \frac{dz}{d\bar{z}}\right) \right|^2 |dz| \quad .$$

In particular, $d\hat{H}(\Phi)/dt \leq 0$ and Φ is a steady state of (4.18) if and only if $\mathbf{Y}\Phi^{-1}\mathbf{X}^{-1}$ is a quasi-rigid deformation of Σ into Σ_1 .

Remark 5.1. *In case the surfaces Σ, Σ_1 are closed (e.g. both homeomorphic to the sphere \mathbb{S}^2), then $\psi^{(t)} = \psi_0^{(t)}$ as given in (5.2), and there is no need of the component $\psi_1^{(t)}, \psi_2^{(t)}$ which take care of the boundary.*

6 Proofs

Proof. (of Theorem 1) Consider an orbit $\Phi^{(t)}$ induced by $S^{(t)} \in O(\Omega) := O(\Omega; \Omega)$ via

$$\Phi^{(t)} = \Phi^{(0)} \circ S^{(t)} .$$

A tangent of this orbit is given by the left representation

$$\dot{S} = \mathbf{v} \circ S , \mathbf{v} \in o(\Omega) , \quad S^{(0)} = \mathbf{I} . \quad (6.1)$$

Since $\nabla \cdot \mathbf{v} \equiv 0$ by definition it follows by (2.2) that $\det(DS^{(t)}) = \det(G_{S^{(t)}}) \equiv 1$ along this orbit. In particular.

$$\det \left(G_{\Phi^{(0)} \circ S^{(t)}} \right) = \det(G_{\Phi^{(0)}})_{(S^{(t)})} \cdot \det(G_{S^{(t)}})_{(z)} = 1$$

so $\Phi^{(t)} \equiv \Phi^{(0)} \circ S^{(t)} \in O(\Omega; \Omega_1)$. In addition

$$D\dot{S} = [D\mathbf{v} \circ S] DS . \quad (6.2)$$

In the left representation $\Phi^{(t+\tau)} = \Phi^{(t)} \circ S^{(\tau)}$, $S^{(\tau)}(x) = x + \tau \mathbf{v}(x) + O(\tau^2)$, so

$$D\Phi^{(t+\tau)} = \left[D\Phi^{(t)} \circ S^{(\tau)} \right] DS^{(\tau)} = D\Phi^{(t)} + \tau D\Phi^{(t)} D\mathbf{v} + \tau D^2\Phi^{(t)} \mathbf{v} + O(\tau^2) , \quad (6.3)$$

We obtain from (2.8) and (6.3)

$$\begin{aligned} \hat{H}(\Phi^{(t+\tau)}) &= \hat{H}(\Phi^{(t)}) + \tau \int_{\Omega} \text{tr} \left(D(\Phi^{(t)}) (D\mathbf{v} + D^*\mathbf{v}) D^*\Phi^{(t)} \right) dx dy \\ &\quad + \tau \int_{\Omega} \text{tr} \left(\left[D^2(\Phi^{(t)}) D^*\Phi^{(t)} + D\Phi^{(t)} DD^*\Phi^{(t)} \right] \mathbf{v} \right) dx dy + O(\tau^2) . \end{aligned} \quad (6.4)$$

However

$$\left[D^2(\Phi^{(t)}) D^*\Phi^{(t)} + D\Phi^{(t)} DD^*\Phi^{(t)} \right] = D \left| D\Phi^{(t)} \right|^2$$

and integration by parts implies

$$\int_{\Omega} D \left\{ \text{tr} \left(\left| D\Phi^{(t)} \right|^2 \right) \right\} \mathbf{v} dx dy = - \int_{\Omega} \text{tr} \left(\left| D\Phi^{(t)} \right|^2 \right) (\nabla \cdot \mathbf{v}) dx dy = 0 , \quad (6.5)$$

since \mathbf{v} is divergence free. From (6.4, 6.5) we obtain

$$\frac{d}{dt} \hat{H}(\Phi^{(t)}) = \int_{\Omega} \text{tr} \left(D(\Phi^{(t)}) [D\mathbf{v} + D^*\mathbf{v}] D^*(\Phi^{(t)}) \right) dx dy \quad (6.6)$$

Next, note that if Ω is simply connected, then any smooth divergence free vector field can be written as

$$\mathbf{v} = \nabla^{\perp} \psi := (-\psi_y, \psi_x), \quad \psi \in C^{\infty}(\Omega) , \psi \equiv 0 \quad \text{on } \partial\Omega . \quad (6.7)$$

It follows that

$$\frac{1}{2} (D\mathbf{v} + D^*\mathbf{v}) = \frac{1}{2} \mathbf{S}(\psi) + \mathbf{U}(\psi) \quad (6.8)$$

where, using $\square := \partial_x^2 - \partial_y^2$,

$$\mathbf{S}(\psi) := \begin{pmatrix} 0 & \square\psi \\ \square\psi & 0 \end{pmatrix} \quad , \quad \mathbf{U}(\psi) := \begin{pmatrix} -\psi_{x,y} & 0 \\ 0 & \psi_{x,y} \end{pmatrix} . \quad (6.9)$$

Note that, with $\Phi^{(t)} := (\phi_1^{(t)}, \phi_2^{(t)})$,

$$D\Phi^{(t)} D^* \Phi^{(t)} \equiv \left| D\Phi^{(t)} \right|^2 = \begin{pmatrix} \left| \Phi_x^{(t)} \right|^2 & \Phi_x^{(t)} \cdot \Phi_y^{(t)} \\ \Phi_x^{(t)} \cdot \Phi_y^{(t)} & \left| \Phi_y^{(t)} \right|^2 \end{pmatrix} ,$$

so

$$\text{tr} \left(D\Phi^{(t)} \mathbf{S}(\psi) D^* \Phi^{(t)} \right) = 2 \left(\Phi_x^{(t)} \cdot \Phi_y^{(t)} \right) \square\psi \quad (6.10)$$

while

$$\text{tr} \left(D\Phi^{(t)} \mathbf{U}(\psi) D^* \Phi^{(t)} \right) = \left(\left| \Phi_y^{(t)} \right|^2 - \left| \Phi_x^{(t)} \right|^2 \right) \psi_{xy} . \quad (6.11)$$

Hence,

$$\text{tr} \left(D(\Phi^{(t)}) [D\mathbf{v} + D^*\mathbf{v}] D^*(\Phi^{(t)}) \right) = \left(\Phi_x^{(t)} \cdot \Phi_y^{(t)} \right) \square\psi + \left(\left| \Phi_y^{(t)} \right|^2 - \left| \Phi_x^{(t)} \right|^2 \right) \psi_{xy} . \quad (6.12)$$

We now return to the complex notation. From (3.4, 3.5),

$$\left(\Phi_x^{(t)} \cdot \Phi_y^{(t)} \right) \square\psi + \left(\left| \Phi_y^{(t)} \right|^2 - \left| \Phi_x^{(t)} \right|^2 \right) \psi_{xy} = 4\Im(f_{\Phi^{(t)}} \partial_{\bar{z}}^2 \psi) , \quad (6.13)$$

hence, by (6.6) and (6.12, 6.13),

$$\frac{d}{dt} \hat{H} \left(\Phi^{(t)} \right) = -2\Re \left(\int_{\Omega} f_{\Phi^{(t)}} \partial_{\bar{z}}^2 \psi dz \wedge d\bar{z} \right) \equiv 4\Im \left(\int_{\Omega} f_{\Phi^{(t)}} \partial_{\bar{z}}^2 \psi dx dy \right) .$$

□

Proof. (of Proposition 3.1):

We imply integration by parts on (3.6) to obtain

$$\frac{d}{dt} \hat{H} \left(\Phi^{(t)} \right) = 4\Im \left(\int_{\Omega} \int \psi \partial_{\bar{z}}^2 f_{\Phi^{(t)}} dx dy \right) + 2\Re \left(\int_{\partial\Omega} (\psi \partial_{\bar{z}} f_{\Phi^{(t)}} - f_{\Phi^{(t)}} \partial_{\bar{z}} \psi) dz \right) .$$

However, $\psi \equiv 0$ on $\partial\Omega$ by (6.7) so

$$\frac{d}{dt} \hat{H} \left(\Phi^{(t)} \right) = 4\Im \int_{\Omega} \int \psi \partial_{\bar{z}}^2 f_{\Phi^{(t)}} dx dy - 2\Re \int_{\partial\Omega} f_{\Phi^{(t)}} \partial_{\bar{z}} \psi dz . \quad (6.14)$$

Part (i) follows from the first integral (6.14), since ψ is arbitrary in the interior of Ω . Since $\psi \equiv 0$ on $\partial\Omega$, it follows by Corollary 3.1 that $i\partial_{\bar{z}}\psi d\bar{z}$ is real valued on $\partial\Omega$. Hence

$$\Re(f_{\Phi} \partial_{\bar{z}} \psi dz) = -\Im \left(f_{\Phi} \frac{dz}{d\bar{z}} i \partial_{\bar{z}} \psi d\bar{z} \right) = \pm \partial_n \psi \Im \left(f_{\Phi} \frac{dz}{d\bar{z}} \right) . \quad (6.15)$$

Since $\partial_n \psi$ is arbitrary on $\partial\Omega$ we obtain part (ii). □

Proof. (of Lemma 4.3): Let $S, S_1 \in O_\eta(\Delta)$. Then $D(S \circ S_1) = DS_{(S_1)} \circ DS_1$. Hence $\det(D(S \circ S_1)) = \det(DS)_{(S_1)} \det(DS_1)$. By definition,

$$\det(DS) = \eta/\eta(S) \quad ; \quad \det(DS)_{(S_1)} = \eta(S_1)/\eta(S \circ S_1)$$

so $\det(D(S \circ S_1)) = \eta/\eta(S \circ S_1)$, so $S \circ S_1 \in O_\eta(\Delta)$. The same argument holds for $\Phi \circ S$ where $\Phi \in O(\Sigma, \Sigma_1)$. \square

Proof. (of Theorem 2):

We repeat the proof of Theorem 3 up to (6.4). Here (6.4). is replaced by

$$\begin{aligned} \hat{H}_\Delta(\Phi^{(t+\tau)}) &= \hat{H}_\Delta(\Phi^{(t)}) + \tau \int_\Delta \mu^2(\Phi^{(t)}) \operatorname{tr} \left(D(\Phi^{(t)}) (D\mathbf{v} + D^*\mathbf{v}) D^*\Phi^{(t)} \right) dx dy \\ &\quad + \tau \int_\Delta \mu^2(\Phi^{(t)}) \operatorname{tr} \left(\left[D^2(\Phi^{(t)}) D^*\Phi^{(t)} + D\Phi^{(t)} D D^*\Phi^{(t)} \right] \mathbf{v} \right) dx dy + \\ &\quad \tau \int_\Delta \left(\nabla \mu_{(\Phi^{(t)})}^2 \cdot \mathbf{v} \right) \operatorname{tr} \left(|D\Phi^{(t)}|^2 \right) dx dy + O(\tau^2) . \end{aligned} \quad (6.16)$$

As in (6.5), the third term on the right of (6.16) is reduced to

$$\int_\Omega \mu^2(\Phi^{(t)}) D \left\{ \operatorname{tr} \left(|D\Phi^{(t)}|^2 \right) \right\} \mathbf{v} dx dy = - \int_\Omega \operatorname{tr} \left(|D\Phi^{(t)}|^2 \right) \nabla \cdot (\mu^2(\Phi^{(t)}) \mathbf{v}) dx dy . \quad (6.17)$$

Together with the forth term in (6.16) we obtain

$$\begin{aligned} \hat{H}_\Delta(\Phi^{(t+\tau)}) &= \hat{H}_\Delta(\Phi^{(t)}) + \tau \int_\Delta \mu^2(\Phi^{(t)}) \operatorname{tr} \left(D(\Phi^{(t)}) (D\mathbf{v} + D^*\mathbf{v}) D^*\Phi^{(t)} \right) dx dy \\ &\quad - \tau \int_\Delta \mu_{(\Phi^{(t)})}^2 (\nabla \cdot \mathbf{v}) \operatorname{tr} \left(|D\Phi^{(t)}|^2 \right) dx dy + O(\tau^2) . \end{aligned} \quad (6.18)$$

Since $\mathbf{v} \in o_\eta$ it follows from (4.17) and (6.18)

$$\begin{aligned} \hat{H}_\Delta(\Phi^{(t+\tau)}) &= \hat{H}_\Delta(\Phi^{(t)}) + \tau \int_\Delta \mu^2(\Phi^{(t)}) \operatorname{tr} \left(D(\Phi^{(t)}) (D\mathbf{v} + D^*\mathbf{v}) D^*\Phi^{(t)} \right) dx dy \\ &\quad + \tau \int_\Delta \mu^2(\Phi^{(t)}) \eta^{-1} (\mathbf{v} \cdot \nabla \eta) \operatorname{tr} \left(|D\Phi^{(t)}|^2 \right) dx dy + O(\tau^2) . \end{aligned} \quad (6.19)$$

We now use $\mathbf{v} \sim i\eta^{-1} \partial_{\bar{z}} \psi$ and (6.7-6.8) to obtain

$$D\mathbf{v} + D^*\mathbf{v} = \eta^{-1} \mathbf{S}(\psi) + 2\eta^{-1} \mathbf{U}(\psi) + 2\eta^{-2} \begin{pmatrix} -\eta_x \psi_y & \frac{\eta_x \psi_x - \eta_y \psi_y}{2} \\ \frac{\eta_x \psi_x - \eta_y \psi_y}{2} & \eta_y \psi_x \end{pmatrix} .$$

By direct calculation

$$\begin{aligned} 2\mu^2(\Phi^{(t)}) \eta^{-2} \operatorname{tr} \left\{ \begin{pmatrix} -\eta_x \psi_y & \frac{\eta_x \psi_x - \eta_y \psi_y}{2} \\ \frac{\eta_x \psi_x - \eta_y \psi_y}{2} & \eta_y \psi_x \end{pmatrix} \begin{pmatrix} |\Phi_x^{(t)}|^2 & \Phi_x^{(t)} \cdot \Phi_y^{(t)} \\ \Phi_x^{(t)} \cdot \Phi_y^{(t)} & |\Phi_y^{(t)}|^2 \end{pmatrix} \right\} = \\ 2\mu^2(\Phi^{(t)}) \eta^{-2} \left(-\eta_x \psi_y |\Phi_x^{(t)}|^2 + (\eta_x \psi_x - \eta_y \psi_y) \Phi_x^{(t)} \cdot \Phi_y^{(t)} + \eta_y \psi_x |\Phi_y^{(t)}|^2 \right) . \end{aligned} \quad (6.20)$$

while

$$\mu^2(\Phi^{(t)})\eta^{-1}(\mathbf{v} \cdot \nabla \eta) \text{tr} \left(|D\Phi^{(t)}|^2 \right) = \mu^2(\Phi^{(t)})\eta^{-2} (\psi_y \eta_x - \psi_x \eta_y) \left(|\Phi_x^{(t)}|^2 + |\Phi_y^{(t)}|^2 \right) . \quad (6.21)$$

using (6.10,6.11) and (6.18-6.21) and differentiation at $\tau = 0$ yield

$$\begin{aligned} \frac{d}{dt} \hat{H}_\Delta \left(\Phi^{(t)} \right) &= \int_\Delta \mu^2 \left(\Phi^{(t)} \right) \eta^{-1} \left[\left(\Phi_x^{(t)} \cdot \Phi_y^{(t)} \right) \square \psi + \left(\left| \Phi_y^{(t)} \right|^2 - \left| \Phi_x^{(t)} \right|^2 \right) \psi_{xy} \right] dx dy \\ &+ \int_\Delta \mu^2 \left(\Phi^{(t)} \right) \eta^{-2} \left(2(\eta_x \psi_x - \eta_y \psi_y) \Phi_x^{(t)} \cdot \Phi_y^{(t)} + (\eta_y \psi_x + \eta_x \psi_y) \left(|\Phi_y^{(t)}|^2 - |\Phi_x^{(t)}|^2 \right) \right) dx dy . \end{aligned} \quad (6.22)$$

In complex notations (3.4), (6.22) takes the form

$$\begin{aligned} \frac{d}{dt} \hat{H}_\Delta \left(\Phi^{(t)} \right) &= 4\Im \left\{ \int_\Delta \mu^2 \left(\Phi^{(t)} \right) \eta^{-1} f_\Phi \partial_{\bar{z}}^2 \psi dx dy + \int_\Delta \mu^2 \left(\Phi^{(t)} \right) (\partial_{\bar{z}} \eta^{-1}) (\partial_{\bar{z}} \psi) f_\Phi dx dy \right\} \\ &= 4\Im \int_\Delta \mu^2 \left(\Phi^{(t)} \right) f_\Phi \partial_{\bar{z}} (\eta^{-1} \partial_{\bar{z}} \psi) dx dy . \end{aligned} \quad (6.23)$$

□

Proof. (of Proposition 4.1): Integration by parts of (4.19) yields:

$$\begin{aligned} \frac{d}{dt} \hat{H}_\Delta \left(\Phi^{(t)} \right) &= 4\Im \left(\int_\Omega \int \psi^{(t)} \partial_{\bar{z}} \left\{ \partial_{\bar{z}} \left(\mu^2(\Phi^{(t)}) f_{\Phi^{(t)}} \right) \eta^{-1} \right\} dx dy \right. \\ &\quad \left. - 2\Re \left(\int_{\partial\Omega} \eta^{-1} \mu^2(\Phi^{(t)}) f_{\Phi^{(t)}} \partial_{\bar{z}} \psi^{(t)} dz \right) \right) . \end{aligned} \quad (6.24)$$

The rest of the proof is, essentially, the same as that of Proposition 4.1. □

Proof. (of Theorem 3 and Theorem 4): The proof of both Theorems follows from (6.14) and (6.24), respectively, where (6.15) is applied in both cases. □

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